A METHOD OF ANALYZING PERIODIC PROCESSES AND ITS APPLICATION TO PROBLEMS OF HEAT CONDUCTION

Yu. P. Kamaev and N. V. Diligenskii UDC 536.2

A method is given for analyzing periodic processes.

The problem of finding a periodic solution of a linear differential equation arises in the investigation of temperature fields in bodies subject to thermal sources varying periodically in time. The most widely used and highly universal method for solving such problems is that of trigonometric Fourier series, which, however, suffer from the fact that in many cases the series are poorly convergent. Moreover, in the majority of cases representation of the solution in the form of a trigonometric series is not clearly distinguishable and lends itself poorly to a qualitative study.

The aim of the present paper is the formulation of a method which would enable us to find a representation of periodic solutions differing from that of Fourier series, and in those cases in which the periodic solutions admit a representation in finite form in terms of elementary or special functions, our method would enable us to find these representations. At the basis of our method are certain relationships, which tie in with the theory of trigonometric Fourier series.

Consider a periodic function f(t) of period T. On $[0, \infty)$ we construct a function $\varphi(t)$ satisfying the following conditions:

1)
$$\varphi(t) = f(t), \quad 0 \leqslant t < T; \tag{1}$$

2) the function
$$\nu(t) = \varphi(t) - \varphi(t+T) \tag{2}$$

is not identically zero;

3) the Laplace transforms of the function $\varphi(t)$ and $\nu(t)$ exist
$$\bar{\varphi}(p) = \int_0^\infty \varphi(t)\, e^{-pt}\, dt, \tag{3}$$

$$\bar{\nu}(p) = \int_0^\infty \nu(t)\, e^{-pt}\, dt; \tag{4}$$

4) the function $\bar{\varphi}(p)$ has no more than a finite number of singular points coinciding with the points
$$p_n = i\,\frac{2\pi}{T}\,n \,(n = 0, \pm 1, \pm 2, \ldots).$$

These conditions do not require the function $\varphi(p)$ to be uniquely defined on $[T, \infty)$. However in practice it is usually expedient to formulate the function $\varphi(t)$ through an analytic extension of the given function f(t) with the half-open interval $[0, T)$.

V. V. Kuibyshev Polytechnical Institute, Kuibyshev. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 17, No. 3, pp. 541-546, September, 1969. Original article submitted October 8, 1968.

Applying the Laplace transformation to Eq. (2) we obtain

$$\bar{\nu}(p)=(1-e^{pT})\bar{\varphi}(p)+e^{pT}\int_0^T \varphi(t)e^{-pt}\,dt. \tag{5}$$

It is evident from this that for all $p_n = i(2\pi/T)n$, excluding those points coincident with singular points of the function $\bar{\varphi}(p)$, we shall have

$$\bar{\nu}\left(i\frac{2\pi}{T}n\right)=\int_0^T \varphi(t)\exp\left(-i\frac{2\pi}{T}nt\right)dt, \tag{6}$$

or, equivalently,

$$c_n=\frac{1}{2\pi}\bar{\nu}\left(i\frac{2\pi}{T}n\right), \tag{7}$$

where the c_n are complex Fourier coefficients of the function f(t).

Based on the relationships obtained, we now have a method, differing from the classical method of harmonic analysis, consisting of the consecutive satisfaction of the operations defined by Eqs. (2), (4), and (6). Moreover, satisfaction of these operations in the reverse order furnishes a way to solve the inverse problem – the problem of harmonic synthesis.

We shall speak of the function $\nu(t)$, associated with the periodic function f(t) by means of the relations (1), (2) under the conditions (6), as the difference function with respect to the periodic function f(t).

The indicated algorithms for harmonic analysis and harmonic synthesis are at the basis of the method we propose for finding periodic solutions of linear differential equations with constant coefficients. For simplicity we consider below an ordinary differential equation. The results carry over automatically to partial differential equations with boundary conditions periodic in one of the variables.

Along with the usual problem, that of finding a periodic component of the solution of the equation

$$Dy=f(t) \tag{8}$$

(D denotes the operation of differentiation), we consider an auxiliary problem, namely that of finding the complete solution of the equation

$$Dx=\nu(t) \tag{9}$$

for zero initial conditions (Cauchy problem). Here D is the same differential operator as that appearing in Eq. (8), and $\nu(t)$ is the difference function with respect to the periodic function f(t).

With the results obtained above we can then show that under these conditions the function f(t), being a solution of the auxiliary problem of Cauchy, is the difference function with respect to the periodic solution of the basic problem. This is then the main result, permitting us to reduce the problem of finding a periodic solution to the much simpler Cauchy problem with the subsequent solving of an equation of the form (2) with single-valued conditions of the form (6).

We note also that if a solution b(t) of Eq. (9) is known when the right member of Eq. (9) is a δ-function $\delta(t)$, and where we have zero initial conditions, then the periodic solution of Eq. (8) is given by the formula

$$y(t)=\int_0^T f(\tau)\xi(t-\tau)\,d\tau,\quad 0\leqslant t<T, \tag{10}$$

where $\xi(t)$ is a periodic function of period T such that b(t) is a difference function with respect to it.

For a practical computation of the integral (10) it is convenient to represent it in the form

$$y(t)=\int_0^t f(\tau)b(t-\tau)\,d\tau+\int_0^T f(\tau)\beta(t+T-\tau)\,d\tau, \tag{11}$$

where $\beta(t)$ is defined by the equation

$$\beta(t)-\beta(t+T)=b(t) \tag{12}$$

TABLE 1

Formula number	$v(t)$	$\varphi(t)$
1	1	$-\frac{t}{T}+\frac{1}{2}$
2	t	$-\frac{t^2}{2T}+\frac{t}{2}-\frac{T}{12}$
3	t^n	$-\frac{T^n}{n+1}B_{n+1}\left(\frac{t}{T}\right)^*$
4	t^α $\alpha>-1$	$T^\alpha\zeta\left(-\alpha,\ \frac{t}{T}\right)^\dagger$
5	$e^{\alpha t}$	$\frac{e^{\alpha t}}{1-e^{\alpha T}}+\frac{1}{\alpha T}$
6	$t^n e^{\alpha t}$	$\frac{\partial^n}{\partial\alpha^n}\left(\frac{e^{\alpha t}}{1-e^{\alpha T}}+\frac{1}{\alpha T}\right)$
7	$\ln\alpha t$	$-\ln\Gamma\left(\frac{t}{T}\right)-\frac{t}{T}\ln\alpha T+\frac{1}{2}\ln 2\pi\alpha T$

*$B_n(t/T)$ are Bernoulli polynomials.

†$\zeta(-\alpha,\ t\ T)$ is the generalized Riemann zeta-function.

with single-valued conditions of the form (6). In Table 1 we display some solutions of Eq. (2). In forming this table the single-valued conditions (6) were imposed upon the desired function for all $n \neq 0$. For the constant component we used the condition

$$\int_0^T \varphi(t)\,dt=0, \tag{13}$$

which is associated with the fact that fairly often in practice one meets the case in which the point $p_0 = 0$ is a singular point of the functions $\bar{\varphi}(p)$ and $\bar{\nu}(p)$, requiring the solution to be supplemented with a constant component.

We consider several examples. We find an expression for the periodic temperature field in an infinite insulated rod when its end face is subjected to a succession of point sources of period T.

The function for a unit point source for such a rod has the form

$$\vartheta(t,\ x)=\frac{Q}{\sqrt{\pi a t}}\,e^{-\frac{x^2}{4at}}. \tag{14}$$

Expanding $\delta(t)$ in a Fourier series and knowing the solution for each harmonic, we find an expression for the periodic temperature

$$\vartheta_\pi(t,\ x)=\frac{Q\sqrt{2}}{\sqrt{\pi a T}}\sum_{n=1}^{\infty}\frac{e^{-\sqrt{\frac{\pi n}{aT}}\,x}}{\sqrt{n}}\cos\left(\frac{2\pi n}{T}\,t-\sqrt{\frac{\pi n}{aT}}\,x-\frac{\pi}{4}\right). \tag{15}$$

This series is suitable for use when x is not too small, in which case the convergence is rapid. As the surface is approached the series converges but poorly. At the end face the series has the form

$$\vartheta_\pi(t)=\frac{Q\sqrt{2}}{\sqrt{\pi a T}}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{2\pi n}{T}\,t-\frac{\pi}{4}\right)}{\sqrt{n}}. \tag{16}$$

Let us find an expression for the temperature at the end face using the method described in this paper. For this it is sufficient to solve Eq. (12) with the right member defined by $Q/\sqrt{\pi a T}$. Consequently we have (formula 4, Table 1):

$$\vartheta_\pi(t)=\frac{Q}{\sqrt{\pi a T}}\,\zeta\left(\frac{1}{2},\ \frac{t}{T}\right). \tag{17}$$

Naturally, the expressions (16) and (17) are equivalent. We find an expression for the steady-state end-face temperature due to the action of a periodic heat source, defined over one period as follows:

$$Q = \begin{cases} Q, & 0 < t < \alpha, \\ 0, & \alpha < t < T. \end{cases}$$

Integrating Eq. (16) we find the solution in the form of a Fourier series:

$$\vartheta_\Pi(t) = \frac{Q}{\pi} \sqrt{\frac{2T}{\pi a}} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{\pi n \alpha}{T}\right)}{n\sqrt{n}} \cos\left(\frac{2\pi n}{T}\, t - \frac{\pi n \alpha}{T} - \frac{\pi}{4}\right). \tag{18}$$

Convergence of the series (18) is very poor. Calculations carried out on the M-20 digital computer show that at some points 100,000 terms of the series is insufficient to guarantee convergence. To obtain the solution by the method of this paper we need to solve Eq. (2) with the left member equal to

$$\vartheta(t) = 2Q\left(\sqrt{\frac{t}{\pi a}} - \sqrt{\frac{t-\alpha}{\pi a}}\right).$$

We obtain the solution in the form

$$\vartheta_\Pi(t) = 2Q\sqrt{\frac{T}{\pi a}} \begin{cases} \zeta\left(-\frac{1}{2}, \frac{t}{T}\right) - \zeta\left(-\frac{1}{2}, \frac{t-\alpha+T}{T}\right), & 0 < t < \alpha, \\ \zeta\left(-\frac{1}{2}, \frac{t}{T}\right) - \zeta\left(-\frac{1}{2}, \frac{t-\alpha}{T}\right), & \alpha < t < T. \end{cases} \tag{19}$$

Integral representations and power series are available for evaluating the generalized ζ-function.

Consider now a two-dimensional adiabatic problem. We find an expression for the periodic component of the temperature field in the neighborhood of a point source. The source strength is a periodic function of the time; over the period [0, T] it grows linearly with the time:

$$Q(t) = Qt, \quad 0 < t < T.$$

The difference function is a constant,

$$\nu(t) = -QT,$$

and the periodic solution of the problem due to a constant is well known [1] and for $r^2/4aT < 0.01$ is equal to

$$\vartheta(t) = -\frac{QT}{4\pi a} \ln\frac{4at}{r^2} + \frac{\gamma QT}{4\pi a}. \tag{20}$$

The solution of the problem stated is obtained by solving Eq. (2) with the left member (20):

$$\vartheta_\Pi(t) = \frac{QT}{4\pi a}\left[\ln\Gamma\left(\frac{t}{T}\right) + \frac{t}{T}\left(\ln\frac{4aT}{r^2} - \gamma\right) - \frac{1}{2}\ln\frac{8\pi aT}{r^2} + \frac{\gamma}{2}\right]. \tag{21}$$

LITERATURE CITED

1. H. S. Carslaw and J. C. Jaeger, Conduction of Heat in Solids, Clarendon Press, Oxford (1959).
2. A. M. Zaezdnyi, Harmonic Synthesis in Radio Engineering and Communication Theory [in Russian], GÉI (1961).
3. A. O. Gel'fond, Calculus of Finite Differences [in Russian], Nauka (1967).
4. N. V. Diligenskii and Yu. P. Kamaev, Compendium of Papers on Heat and Mass Transfer [in Russian], Vol. 8, Minsk (1968).